COORDINATION OF GEOMETRY REPRESENTATIONS IN A TEXTBOOK

Waldemar Stepnowski
Temple University
wstepnowski@temple.edu

The study describes how the common representations are coordinated. I analyzed a popular geometry textbook using semiotics and a pragmatic approach to capture the variety of representations into categories and to use descriptive statistics to narrow the focus to the most common representations and coordinations. The major findings are: (1) exposing which representations are most often coordinated like written language; (2) some of the mechanisms in coordination use numbers, point names, and textbook gestures, which include color, arrows, font changes, etc.

Keywords: Geometry and Spatial Reasoning, High School Education

Introduction

Some representations are ideal in certain situations, while some are not. Teachers, students, and seasoned mathematicians must make choices in how they represent a mathematical object so that it helps them attain their goal. To be able to make choices in the use of geometry representations, teachers, students, and mathematicians need to know which representations are available and how to use them. That is precisely the goal of my research, i.e., to identify how they are presented and coordinated in a high school geometry textbook, Larson and Boswell's (2015) *Geometry: A Common Core Curriculum*. I do that by analyzing the mechanisms that aid in the most common coordinations.

Besides the above, there are a few more reasons why studying representations is valuable. First, we all want students to solve problems with a high degree of accuracy and efficiency, and representations help us manage mathematical information. Students who coordinate different mathematical representations well solve mathematical problem better than those who do not (Gagatsis & Shiakalli, 2004). Second, there is a difference between changing within one representation and changing or coordinating between different representations. While students have difficulty changing representations (conversion), they have less difficulty in manipulating (treatment) a given representation (Duval, 2006). Therefore, studying which representations appear to be coordinated most often and what are some ways that they are coordinated may help geometry teachers, publishers, and researchers devise strategies that aid students with those conversions. Third, to solve problems students need to use aids or tools like representations to reduce the complexity of a problem. We use representations as a kind of external memory, where we can offload some of the information from our working memory (Scaife & Rogers, 1996). Chandler and Sweller's (1991) seminal work on cognitive load theory shows via experiments that integrated diagrams with text, in contrast to a diagram with text below it, reduced the cognitive load, and reducing cognitive load improves learning. Fourth, we are beginning to develop digital textbooks, online environments, etc., where the choice of representations is important (Presmeg etal., 2016).

Framework

I take the formalist tradition as to the ontology of what mathematics is, and the pragmatist tradition as to the epistemology of this study. Cognitive Load theory was the learning theory that guided how I analyzed how the signs of geometry could be learned. According to cognitive load

theory, students make use of all three components of working memory, namely central executive, visual-spatial sketchpad and an auditory loop (Sweller, Ayres, & Kalyga, 2011). Our working memory is limited to a sentence, maybe two, an equation or two, a small diagram, a few ordered pairs, etc. As we will see in the methods, a sentence, or more specifically an independent clause, was my unit of analysis.

Semiotics also provides us some tools to analyze representations of mathematical objects. In Peirce's semiotics, the object is what is signified, the representation is the signifier or representamen, and a student's understanding of the object is the interpretant (Sáenz-Ludlow, 2002; Schreiber, 2013). Sáenz-Ludlow (2002) points out that representations are a process (semiosis) among the object, the representamen, and the interpretant. In the case of mathematics, there is little or no direct contact with the object, and therefore a student coordinates representamen ('symbols') and interpretants (students' understanding of the object) to understand the properties of the object or even the object itself.

Converting from one representation to another is difficult for students. Students studying geometry transformations, for example, preferred to use algebraic transformation rules that they memorized rather than think visually (Bansilal & Naidoo, 2012). When student do convert, they are better at it in one direction than the other. Expert teachers in China recognize the importance of converting representation by having students change language to geometrical symbol language, text or oral statements to diagrams, diagram and/or geometrical symbol language to verbal description (Ding, Jones, & Zhang, 2013). The difference between expert and novice may be best explained with Cognitive Load Theory (CLT). Sweller, Ayres, Kalyuga (2011) explain that the difference between a novice and an expert is that an expert has biologically secondary information (geometry representations, algorithms, etc.) stored in his or her long-term memory.

Methods

I studied a high school geometry textbook and focused on the most common coordinations to expose any common mechanisms that aid in coordination. The textbook in this study is Larson and Boswell's (2015) *Geometry: A Common Core Curriculum.* I chose to study the chapter one (Basics of Geometry), chapter 2 (Reasoning and Proofs), chapter 3 (parallel and perpendicular lines), chapter 4 (Transformations), and most of chapter 11 (Circumference, Area, and Volume), because all the representations were either introduced in those chapters or used extensively enough to study. To determine that selection I had to analyze representations thoroughly in the first chapter, where most representations were introduced.

From multiple initial passes through the textbook, I found many possible distinctions of how objects and concepts are represented in geometry. I also noticed that many chapters used the same representations with little variation (mostly written language and diagram). I excluded some chapters and focused on deeper analysis of coordination and the mechanisms that aid that coordination in chapters 1, 2, 3, 4, and 11. I did not analyze the exercises at the end of the sections and chapters, the preface, index, appendix, glossary, etc. Instead, I analyzed the expository text, postulates/theorems, worked examples, explorations, constructions, etc., that are meant to introduce students to new mathematical content.

I coded almost 4,000 coordinations into a spreadsheet, giving me the ability to look for common types of coordinations like written language to diagram (WL→D). Having those coordinations, I looked for common mechanisms that could aid in the coordination. This was not a linear process, meaning I had to return to previous stages to refine data, definitions, coordinations, etc. I printed each page of the textbook sometimes multiple times and cut out each independent clause and any associated representations like graphs, diagrams, algebraic

expressions, etc. I labeled each snippet with the representations present, and I analyzed in which order they could be read by a student. Some aspects had to be operationalized. For example, I had to decide that for a code to be real-world written language (WDR), the meaning of the sentence and most of the words had to reflect the physical real-world. I used the following codes: written language (WL) with sub-codes of real (R), pure (P), meta (M), declarative (D) and imperative/interrogative (I); diagrams (D) with sub-codes of 2D, 3D, construction (C), dynamic geometry environment (E), and graphs (G); algebra (A); short geometry symbols (Sy), numbers (N); physical objects (P); table (T).

Finally, I focused on the mechanisms. O'Halloran's (2008) linguistic approach, along with Peirce's semiotics, aided to understand some of the mechanism that occur during coordination of representations. By mechanism, as I explained earlier, I mean the method through which we change within one representation or convert one representation to another.

Results

I found that there are a few common mechanisms to aid coordination like names of points, numbers, algebra, and textbook gestures like color, font change, arrows, etc. The descriptive statistics of the most common representations or combinations of representations along with some examples are shown on the table below. Some representations like diagrams were counted multiple times because other representations referred to them multiple times. Initial representation means that I considered that representation leading to another, the subsequent representation. Table 1 lists combinations of representations that appeared more than 100 times in the coding. The statistics of the combinations of codes helped me choose which coordination of representation combinations to analyze in more detail; it is more pragmatic to study those combinations of representations that students are most likely to come across.

Table 1: Descriptive statistics of the frequency of representations during coordination

Representation Combination	Initial	Subsequent	Total	Example (frequency)
WL	1521	797	2318	WL → WL (471), WL → D (257)
D	258	844	1102	NOP → D (131), D → WL (86)
WLSy	471	154	625	WLSy → D (231), WL → WLSy (59)
N	141	243	384	NA → N (55), N → N (51)
WLN	181	182	363	WL → WLN (46), N → WLN (36)
NOP	204	123	327	NOP → NOP (42), OPA → NOP (28)
DN	78	121	199	DN → NSy (21), DN → N (16)
NSy	98	101	199	D → NSy (15), DNP → NSy (12)
A	118	78	196	A → WLA (27), A → A (21)
DP		179	179	WL → DP (89), WLSy → DP (49)
NA	115	61	176	NA → WLN (15), NA → A (10)
WLA	76	98	174	WL → WLA (44), A → WLA (27)
Sy	90	55	145	Sy → D (30), Sy → WL (27)
WLD		118	118	WL → WLD (95)
ASy	67	40	107	ASy → WL (25), ASy → ASy (13)

I have chosen to discuss the following coordinations because they appear often in the textbook. It is obviously only a small selection chosen for brevity. The following examples are not exactly what I have as the most common coordinations, but I tried to choose a representative sample that was not too large and, at the same time, covers most of the common coordinations.

WL → WL

The two sentences below demonstrate two concepts in language that are in proximity to each other.

> A *straightedge* is a tool that you can use to draw a straight line. An example of a straightedge is a ruler. (p. 11, Larson & Boswell, 2015)

Both clauses are declarative written language (WDP), and ‘*straightedge*’ is emphasized with italics, a textbook gesture (Gst). The gesture seems to underscore the word ‘straightedge,’ in a similar way a teacher would point to it with her finger as gesture emphasizing the word’s importance. Other important vocabulary words have a yellow background, and the font is in bold. With the italics, a textbook gesture (Gst), we coordinate these two sentences focusing not only on *straightedge* because it is the subject of the first sentence and part of the noun phrase of the second one, but also because it is emphasized through a gesture. Gestures, repeated use of words, and sentence structure (like definitions) aid in connecting representations through emphasis and an easy way to refer to an object when scanning for it.

WDP+Sy → P+2D

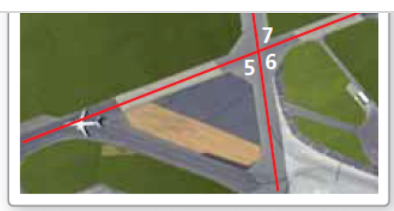
The picture of airport runways with the red lines and angles (2D) is an example of how physical objects (P) can be used to represent mathematical objects. The diagram overlaps the runways, which are clearly wider than a line, and for that reason a thinner line must be drawn somewhere on the broad runway; the authors chose to draw it in the middle of the runway. This runway example shows the problem with using physical objects (P) to represent mathematical ones; it is more difficult to discuss the abstract features of geometry when coordinating with physical objects than diagrams alone.

STUDY TIP

In paragraph proofs, *transitional words* such as *so*, *then*, and *therefore* help make the logic clear.

Given $\angle 5$ and $\angle 7$ are vertical angles.

Prove $\angle 5 \cong \angle 7$



Paragraph Proof

$\angle 5$ and $\angle 7$ are vertical angles formed by intersecting lines. As shown in the diagram, $\angle 5$ and $\angle 6$ are a linear pair, and $\angle 6$ and $\angle 7$ are a linear pair. Then, by the Linear Pair Postulate, $\angle 5$ and $\angle 6$ are supplementary and $\angle 6$ and $\angle 7$ are supplementary. So, by the Congruent Supplements Theorem, $\angle 5 \cong \angle 7$.

Figure 1. WMP, syllogism, Sy, Picture of Physical object (PP). (p. 110, Larson & Boswell, 2015).

In Figure 2, a graph (G) is used to display some of the information in a worked example. To remind the reader, the number of representations for table 3 was reduced; graph (G), construction (C), 2D, and 3D are grouped under one representation diagrams (D).

Determine which of the lines are parallel and which of the lines are perpendicular.

SOLUTION

Find the slope of each line.

Line a: $m = \dfrac{3-2}{0-(-3)} = \dfrac{1}{3}$

Line b: $m = \dfrac{0-(-1)}{2-0} = \dfrac{1}{2}$

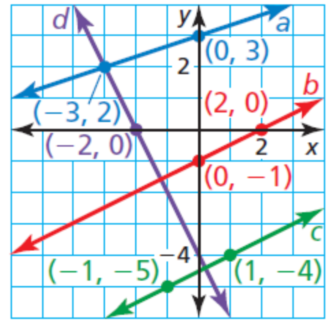


Figure 2. Example of WIP, G, OP, and N combination (p. 157, Larson & Boswell, 2015).

We can notice many features in figure 2: use of color for different objects, ordered pairs, points, grid, short segment (gesture) that points to a point (see point at (-3, 2)). Other graphs have other aspects: equations, variables in ordered pairs, angle names and measures. Figure 2 and the next two paragraphs explain how ordered pair with numbers (NOP) were coordinated with written language and arithmetic of numbers.

WDP+N→G+N+OP

As in figure 2, numbers (N) and ordered pairs (OP) are often coordinated. In this case, 'Line *a*' (WL) and the substituted x- and y-values (N) would be coordinated with the graph (G) and the ordered pairs (NOP). Line *a* is not only labeled with the letter '*a*' on the graph but it is also colored blue for easier coordination. Again, such mechanisms are used often in the book, especially when a concept is being explained for the first or second time. A student would need to have the schema of slope (change of y divided by change in x) well established to follow the worked example, or he or she would need to look back at the formula. In this figure, the student would look back and forth possibly a few times to see that the y-values are 3 and 2 (numerator), and that the x-values (denominator) are 0 and -3. Because we are comparing slopes of various lines to see if they are parallel the lines are differentiated by color; when slope was reviewed earlier in the textbook (p. 123) the change in the x- and y-values was color-coded in a worked example.

The most common mechanism in coordinating objects is using the point. Segments, lines, rays, circles, intersections, polygons, etc., are displayed on diagrams (D) containing named points. They are then referred to in written language (WL) or in short geometry statements (Sy) using those points, that is a student needs to look at specific points on a diagram (D) to see where an object is. For example, a student would look for the three points A, B, and C of △ABC (Sy) on the diagram.

Other mechanism between representations are variables, color, measurement (N), numerals/letters for angles, gestures, and other symbols in statements. Variables or even algebraic expressions (A) often appear as measures of angles or length and then in a paragraph (WL) or statement (Sy, and maybe Sy) near the diagram. Students can then use those variables to coordinate between the two representations. Numbers (N) are also used in a similar manner, often appearing as length or angle measure both on the diagram and in written language and/or statements. Numbers are also crucial in coordination between graphs, points as ordered pairs (NOP), and written language. Students can not only refer to numbers on the axes, but they can also count the squares on the graph to determine how many units a point, line, center, etc., is

away from the x- or y-axis. Numerals and cursive lower-case letters also help with coordination between representations for angles and lines, respectively. Finally, other symbols (Sy) like Δ, $\angle$, $\|$, $\overline{AB}$ (the bar above AB), $\cong$, etc., aid in coordination. Many are iconic, visually close to the objects, but others like the congruence symbol ($\cong$) are arbitrary symbols.

Many words convey meaning in geometry through being connected to physical reality, a mechanism that aids in coordination. That connections may aid in students remembering the word. For example, a word like ray is usually thought in connection to light. In common language, a base is usually defined as the bottom part of some structure like bases in triangles; they appear at the bottom of a corresponding altitude. There are other words from the textbook that mean something in the physical world that is different in varying degrees to what they mean in geometry, e.g., line, compass, construction, equal, side, vertical, adjacent, reflect, intersection, slope, etc.

Movement is not often discussed in geometry class; geometry is usually static, but many of its concepts involve motion. As I analyzed the language, diagrams, constructions, and other representations, movement seems to pervade them. There are many concepts that movement appears in. Dynamic geometry and construction obviously involve motion to construct, and with DGE to drag even after construction. Slope is often described as "rise over run," which implies moving up and then over; even the purer mathematical language of "change in y" implies upward or downward movement. On graphs arrows often show that to move from some point A to some point B, you need to go up a certain number of units and over a certain number of units. Roads or airplane paths, which are superimposed on diagrams, often contain cars or planes (see fig. 1) that are implied to be moving from one point to another. So, when a student thinks of a point or an intersection, he or she might think of a car or a plane passing through it. As another example, term 'bisect' is defined as "cut into … parts," which implies a kind of slicing of a segment or an angle.

Addressing these depictions of movement in geometry concepts, we must consider that formal definitions, concepts, properties, etc. are static sets of points. For example, a line contains a point, not passes through it; a reflection is the set of points the same distance away on the other side of a line, not a flipping of a figure; a line bisects a segment when one of line's points is the midpoint of the segment, not a cutting or a slicing of a segment like a knife. These metaphors may aid students to remember and to connect concepts, or they may confuse them. The point is that representations may complement, contradict, mirror each other, and that is part of the mechanism that helps us coordinate between representations. For example, if a student understands bisecting a segment as cutting into two equal parts, then he or she may think that the segment like a piece of rope is not whole but rather two separate smaller segments because when he or she cuts a segment of rope in half, the piece of rope is no longer whole. On the other hand, from similar experiences, the student's idea of equal parts of a piece of rope may help him or her to remember that bisect means exactly in half.

Discussion

Analysis of geometry representations is very complex in many ways. First, there are a variety of representations. Second, there are a great number of mathematical objects throughout a typical geometry textbook and course; the objects can be typical like points and segments, they can be aspects of those objects like length, they can be relationships like the ratio between two lengths, they can be more general like explanations of the logic of a mathematical system, etc. Third, there are a great number of combinations of the various representations. My research tried to

clarify this specific field of study on the one hand by making specific distinctions among representations; that was accomplished by studying how and which representations were introduced in the textbook (first research question). Then, I tried to simplify the complexity by focusing on the most common coordinations (second research question). Finally, while analyzing those coordinations, I looked for common mechanisms (third research question) that aided in the coordination between or among representations.

The analysis of the introduction of representations, especially the introduction of their affordances, shows that certain representations like written language and graphs are not introduced with detail and with explicitness, while others like dynamic geometry environments (DGE) have more sections devoted to explanations of use and affordances. Yet, the textbook very rarely provides the constraints of specific representations. Having the descriptive statistics and knowing what coordinations of representations that students are exposed to in a geometry course most often, I was able to focus on some mechanisms in high school geometry. For example, coordination of language, diagrams, etc. in many different combinations is very useful, but coordination of a table of values and a graph is not because it does not appear often in the textbook.

Interpretations and Implications

This study explored the introduction of representations in a textbook, but to do that I needed to make distinctions among those representations and go beyond those found in the extant literature. Therefore, while the representations were introduced, I was searching for specific differences between the ones that appeared earlier in the textbook with those that followed, and also with those in the extant literature. If they were similar the ones that came before, there was no introduction, if they were different, I analyzed the introduction. In this section, I would like to compare the representations introduced in the textbook with those in the extant literature, describe how they were introduced, and describe possible implications of both.

It is helpful to discuss coordination with Duval's distinction of treatment and conversion, and with Peirce's triad of object, representamen, and interpretant. In this textbook examination, most coordination involved conversion, where different representations were compared. Some coordinations (about 10%) were treatments, or coordinations within a representation, but most of it was within written language and a small amount of algebra. It was possible to read an implied reference to Peirce's triad where the authors discuss "interpreting a diagram," and list assumptions that a student can and cannot conclude. For example, we cannot conclude that two segments are congruent because they appear to be. This is the closest that the authors come to making clear that the reader has a role to play in the interpretation of representations. There is a representation (representamen) of two segments that appear to be the same length. There is an interpretant, where a student might think that they are or are not congruent. There is the object that whoever designed the diagram for an exercise or a worked example either planned them to be congruent or not, and that the math community has a specific idea of a congruent segments. Neither idea was discussed explicitly even with more teenage-friendly vocabulary in the textbook. The closest explanation of conversion is the use of 'words' and 'symbols' next to those different representations of an object. Again, the implication for future research is to study whether such descriptions of Peirce's triad or Duval's treatment/conversion distinction, or similar ideas more suitable for adolescents in high school geometry, improves student learning. The distinction might provide students with meta-language to process their learning and understanding, which in turn might improve their self-regulating techniques.

From my perspective as a teacher, I found some mechanisms that aided coordination (the third research question). One was using colors to compare relative parts like x- and y-values in ordered pairs and in equations or on a graph, new arcs on construction, highlighting specific words, etc. Another was sentence structure like conditional (if/then) statements or definitions to aid students connect ideas in a familiar way. A third was using textbook gesture like arrows to point to specific locations or parts or to show a transformation or other movement. Fourth, algebra has its refined mechanisms. Fifth, many of the short geometry symbols like parallel (||) are iconic so students can see a smaller version in the flow of text or as separate statements. Sixth, the names of points are the most traditional way of coordinating among representations.

Textbook gestures seem to overlap with mechanisms and studying them more may change how we use them in the classroom, in future research, and in textbooks. First, studying whether they increase cognitive load may warrant using them less or more. For example, having multiple lines on the same diagram as figure 2, color as a textbook gesture may reduce the cognitive load, but there is probably a point when the number of lines overload a student working memory. What is the best number when introducing a new topic like slope of parallel lines? Two? Three? More? For which students, more advanced less? Where is the most practical balance? Another important question is whether the color itself is distracting. It may be that displaying those three or four lines would reduce the load or displaying them in black may reduce load. With digital textbooks being more and more common a third way is possible, being able to turn the color off or on, or displaying it before or after displaying the figures in black. A study using pre- and post-tests could help answer some of these questions.

As with the coordinations, this is not an exhaustive list, and it certainly needs to be studied through the eyes of actual high school students seeing some of these representations for the first time or in new light. Many of the mechanisms are what I termed gestures, e.g., color, arrows, type of font, and they appear in both representations, or join them in some way.

Recommendations

Besides the recommendations for further research, teacher training and textbook authors could benefit from this research. Teacher colleges could train teachers to fill the gaps of textbooks that do not address explicit introduction and coordination of representations. Researchers can learn more about whether explicit introduction to the various representations and their affordances and constraints, explanations of conversions among representations, and teacher knowledge about these topics, leads to better student outcomes. We may discover that students learn these concepts through example, or that teachers fill in the gaps in the textbook well enough that explicit explanations may not be necessary in a textbook. With digital textbooks and websites becoming more abundant, the use of animation, sound, and possibly other representations are possible. Not only is it worthwhile to study digital textbooks and websites to find more dynamic mechanisms than the ones in a printed textbook, but I hope that researchers and textbook and website designers attempt to add more and improved mechanisms that make learning coordination clearer and more engaging.

References

Ding, L., Jones, K, & Zhang, D. (2013). Teaching geometrical theorems in Grade 8 using the 'Shen Tou' method: A case study in Shanghai. In L. Fan et al. (Eds), How Chinese Teach Mathematics. Singapore: World Scientific.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. Educational studies in mathematics, 61(1/2), 103-131. doi:10.1007/s10649-006-0400-z

Gagatsis, A., & Shiakalli, M. (2004). Ability to translate from one representation of the concept of function to another and mathematical problem solving. Educational Psychology, 24(5), 645-657. doi:10.1080/0144341042000262953

Larson, R., & Boswell, L. (2015). Geometry: A Common Core Curriculum. Erie, PA: Big Ideas Learning.
Naidoo, J., & Bansilal, S. (2010). Strategies used by Grade 12 Mathematics learners in Transformation Geometry. South African Journal of Education, 32(1), 26-39.
O'Halloran, K. (2008). Mathematical discourse: Language, symbolism and visual images. London, UK: Continuum.
Presmeg, N., Radford, L., Roth, W. M., & Kadunz, G. (2016). Semiotics in Theory and Practice in Mathematics Education. In Semiotics in Mathematics Education (pp. 5-29). Springer International Publishing.
Sáenz-Ludlow, A. (2002). Sign as a process of representation: A Peircean perspective. Representations and mathematics visualization, 277-296.
Scaife, M., & Rogers, Y. (1996). External cognition: how do graphical representations work? International Journal of Human-Computer Studies, 45(2), 185-213. doi:10.1006/ijhc.1996.0048
Schreiber, C. (2012). Semiotic processes in chat-based problem-solving situations. Educational Studies in Mathematics, 82(1), 51-73. doi:10.1007/s10649-012-9417-7
Sweller, J., Ayres, P. & Kalyuga, S. (2011). Cognitive load theory. New York, NY: Springer.